

Exact results for the criticality of quench dynamics in quantum Ising models

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(Received 1 January 2009; revised manuscript received 16 July 2009; published 5 August 2009)

Based on the obtained exact results we systematically study the quench dynamics of a one-dimensional spin-1/2 transverse field Ising model with zero- and finite-temperature initial states. We focus on the magnetization of the system after a sudden change of the external field and a coherent time-evolution process. With a zero-temperature initial state, the quench magnetic susceptibility as a function of the initial field strength exhibits strongly similar scaling behaviors to those of the static magnetic susceptibility, and the quench magnetic susceptibility as a function of the final field strength shows a discontinuity at the quantum critical point. This discontinuity remains robust and always occurs at the quantum critical point even for the case of finite-temperature initial systems, which indicates a great advantage of employing quench dynamics to study quantum phase transitions.

DOI: [10.1103/PhysRevB.80.054404](https://doi.org/10.1103/PhysRevB.80.054404)

PACS number(s): 64.70.Tg, 05.70.Jk, 75.10.Jm

I. INTRODUCTION

The study of quantum phase transitions (QPTs) is a fascinating topic in condensed-matter physics and quantum information science. QPTs distinctively from temperature-driven critical phenomena occur due to the competition between different parameters describing the interactions of the system, and QPTs occur only at zero temperature.¹ In the second-order QPT, the ground state undergoes qualitative changes when an external parameter passes through the quantum critical points (QCPs). The QCPs are characterized by the divergence in the correlation length, which leads to the critical scaling behaviors governed by a class of universal exponents.² In principle, experimental observations of QPTs could be achieved at zero temperature.¹ However, in practice, it is difficult to realize since cooling matter to zero temperature is impossible in any experiment. Hence, recently, the finite-temperature properties of QPT systems begin to attract more attention.³ In their results, the quantum criticality can persist up to a surprisingly high temperature. However, the critical behavior at finite temperature is not exact but a remanent of that at zero temperature. Some divergent physical quantities at the QCP, e.g., the magnetic susceptibility in the one-dimensional spin-1/2 transverse field Ising model (TFIM), become convergent and shifted away from the QCP in the parameter space at finite temperature.²

On the other hand, besides the ground state, the dynamic properties of QPT systems also cause a lot of interests.⁴⁻¹³ Actually, the zero-temperature static properties are naturally and intimately linked to the dynamic process, in which many excited states are involved. As we know, the scaling exponents are dependent on the effective dimensionality, which is the sum of the dimension and the dynamic exponent.^{1,2} For the adiabatic approach of the QPT system to the QCP, the variation in parameter needs to slow down to infinitesimal.⁷ So the ground state at the QCP could not be achieved adiabatically. Differently, quench dynamics can pass through the QCP without any restriction. In experiments, the quench dynamics of the Bose-Hubbard model have been performed with ultracold bosonic atoms in optical lattices.⁸ Quench dy-

namics and their critical behaviors in the TFIM are also investigated theoretically.^{10,11} However, none of them discussed the effect of temperature on these critical behaviors. The quench dynamics relate to all of the eigenstates more than just the ground state, so the quench dynamic properties at the QCP should not be as sensitive to temperature as those of the ground state.

With these motivations, in this paper, we study the quench dynamics of the TFIM with zero-temperature and finite-temperature initial states. Initially, the system is modulated to be with a transverse field strength λ_i and temperature T . The field strength is suddenly changed to λ_f and the system begins to evolve coherently. Here, we use the magnetization per spin in the transverse field direction $m(\lambda_i, \lambda_f, T; t)$ to characterize the state at time t . After a long enough time evolution, the off-diagonal contributions to $m(\lambda_i, \lambda_f, T; t)$ are cancelled with each other,¹⁴ and $m(\lambda_i, \lambda_f, T; t)$ achieves a steady value finally. The final asymptotic magnetization is

$$m_q(\lambda_i, \lambda_f, T) = m(\lambda_i, \lambda_f, T; \infty), \quad (1)$$

which is called the quench magnetization in this paper. To make the above formula more easily tractable mathematically, we write it as¹⁴

$$m_q(\lambda_i, \lambda_f, T) = \lim_{\tau \rightarrow \infty} \int_0^\tau \frac{dt}{\tau} m(\lambda_i, \lambda_f, T; t). \quad (2)$$

We will show that m_q as well as its derivatives, the magnetic susceptibilities $\chi_i(\lambda_i, \lambda_f, T) = \partial m_q / \partial \lambda_i$ and $\chi_f(\lambda_i, \lambda_f, T) = \partial m_q / \partial \lambda_f$, exhibit critical behaviors when λ_i or λ_f passes through the QCP. For a zero-temperature initial state, the susceptibility χ_i diverges logarithmically and exhibits scaling behaviors when λ_i is in the vicinity of the QCP. For an initial state with temperature T , the susceptibility χ_f experiences a jump $\delta\chi_f = \tanh(\Delta_i/2T)$ when λ_f is at the QCP, where Δ_i is the energy gap between the ground and first-excited states of the initial system. The jump $\delta\chi_f$ achieves maximum when T turns to zero. Since this jump is not sensitive to T and the operations are not restricted by adiabatic conditions, the

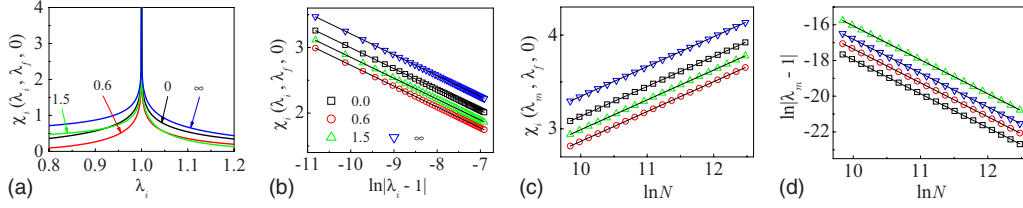


FIG. 1. (Color online) Scaling behaviors of the susceptibility $\chi_i(\lambda_i, \lambda_f, 0)$ with $\lambda_f=0, 0.6, 1.5$, and ∞ . (a) and (b) are plots of $\chi_i(\lambda_i, \lambda_f, 0)$ in the thermodynamic limit. (c) and (d) are plots of $\chi_i(\lambda_m, \lambda_f, 0)$ and $\ln|\lambda_m - 1|$ as functions of $\ln N$, where λ_m is the pseudocritical point. The slopes of the solid lines are $-\pi^{-1}$ in (b), π^{-1} in (c), and -1.9 in (d).

quench dynamic process has the advantage to study critical behaviors of QPT systems without the rigorous restriction of zero temperature.

II. QUENCH MAGNETIZATION

The TFIM is a famous model for studying the second-order QPTs. It is exactly solvable and useful for verifying many new concepts and methods. The Hamiltonian of the TFIM is

$$H(\lambda) = - \sum_{j=1}^N (\sigma_j^x \sigma_{j+1}^x + \lambda \sigma_j^z), \quad (3)$$

where σ^α is the Pauli matrix ($\alpha=x, y, z$) and N is the number of sites. The QCP of this system is at $\lambda=1$. Initially, the strength of the transverse field is λ_i and the system is in a thermal state $\rho(\lambda_i, T; 0) = Z^{-1} \exp[-H(\lambda_i)/T]$ with temperature T , where $Z = \text{Tr} \exp[-H(\lambda_i)/T]$. Then the transverse field is suddenly changed to λ_f and the system begins to evolve coherently driven by $H(\lambda_f)$. The magnetization per spin in the transverse field direction is

$$m(\lambda_i, \lambda_f, T; t) = \text{Tr} \Sigma^x \rho(\lambda_i, T; t), \quad (4)$$

where $\Sigma^x = 1/N \sum_{j=1}^N \sigma_j^x$ and $\rho(\lambda_i, T; t) = e^{-iH(\lambda_f)t} \rho(\lambda_i, T; 0) e^{iH(\lambda_f)t}$ is the state at time t . After a long enough time evolution, m approaches the steady quench magnetization

$$m_q(\lambda_i, \lambda_f, T) = \text{Tr} \hat{m}_q(\lambda_f) \rho(\lambda_i, T; 0), \quad (5)$$

where the quench magnetization operator $\hat{m}_q(\lambda_f) = \sum_n |n(\lambda_f)\rangle \langle n(\lambda_f)| \Sigma^x |n(\lambda_f)\rangle \langle n(\lambda_f)|$. Here, $\{|n(\lambda)\rangle\}$ is the complete set of eigenstates of $H(\lambda)$, i.e., $H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)\rangle$, and $n=0$ corresponds to the ground state.

Using the exact solution of TFIM (Refs. 1, 5, and 15) and taking the thermodynamic limit $N \rightarrow \infty$, we get the quench magnetization as

$$m_q(\lambda_i, \lambda_f, T) = \int_0^{2\pi} \frac{dk}{2\pi} \tanh[\epsilon(\lambda_i, k)/2T] \times \cos \theta(\lambda_f, k) \cos[\theta(\lambda_f, k) - \theta(\lambda_i, k)], \quad (6)$$

where the function $\theta(\lambda, k)$ is defined as $\cos \theta(\lambda, k) = 2(\lambda - \cos k)/\epsilon$, $\sin \theta(\lambda, k) = 2 \sin k/\epsilon$, and $\epsilon(\lambda, k) = 2\sqrt{1 + \lambda^2 - 2\lambda \cos k}$. The quench magnetization is dependent on two systems, the initial system with the transverse field

strength λ_i and the final system with λ_f . In the following, we will consider two cases, $\lambda_i \sim 1$ and $\lambda_f \sim 1$, respectively.

III. SCALING BEHAVIORS

We first consider the case of zero-temperature initial state with $\lambda_i \sim 1$, where $\rho(\lambda_i, 0, 0) = |0(\lambda_i)\rangle \langle 0(\lambda_i)|$. It should be noted that in the thermodynamic limit, the ground state of $H(\lambda_i)$ is twofold degenerate in the region $\lambda_i < 1$. Fortunately, any combination of these two degenerate states as an initial state will give the same result. From Eq. (5) we have $m_q(\lambda_i, \lambda_f, 0) = \langle 0(\lambda_i) | \hat{m}_q(\lambda_f) | 0(\lambda_i) \rangle$, which writes $m_q(\lambda_i, \lambda_f, 0)$ as an expectation value of the operator $\hat{m}_q(\lambda_f)$ in the ground state $|0(\lambda_i)\rangle$ and presumably implies that a sudden change of the ground state $|0(\lambda_i)\rangle$ around $\lambda_i=1$ will lead to some critical behaviors of $m_q(\lambda_i, \lambda_f, 0)$. Actually, in the strong-field limit $\lambda_f = \infty$, the transverse field term is dominant, which keeps the conservation of the expectation value of the total spin component in the transverse field direction during the quench process. Consequently, we have $m_q(\lambda_i, \infty, 0) = m_q(\lambda_i, \lambda_i, 0)$ and $\chi_i(\lambda_i, \infty, 0) = \chi_i(\lambda_i, \lambda_i, 0)$, where $m_q(\lambda_i, \lambda_i, 0)$ and $\chi_i(\lambda_i, \lambda_i, 0)$ are the static magnetization and susceptibility, respectively. $\chi_i(\lambda_i, \infty, 0)$ should share the same critical behaviors as those of $\chi_i(\lambda_i, \lambda_i, 0)$, the second-order derivative of the ground-state energy density. So $\chi_i(\lambda_i, \infty, 0)$ should diverge at $\lambda_i=1$ and obey scaling behaviors at $\lambda_i \sim 1$.

Next we will show that a similar conclusion could be obtained for finite λ_f . Define $\Lambda_i = \min\{\lambda_i, \lambda_i^{-1}\}$ and $\Lambda_f = \min\{\lambda_f, \lambda_f^{-1}\}$. When $\Lambda_i > \Lambda_f$, from Eq. (6) we have¹⁶ $m_q(\lambda_i, \lambda_f, 0) = \int_0^{\Lambda_i} x^{-1} dx (\pi^{-1}v + w\sqrt{\Lambda_i/4x}) + w$, where $v = \text{sgn}(\Lambda_f - x)c(\lambda_f, x)[s(\lambda_f, x)s(\lambda_i, x) - c(\lambda_f, x)c(\lambda_i, x)]$, $c(\lambda, x) = [2\lambda - (x+x^{-1})]/d$, $s(\lambda, x) = (x-x^{-1})/d$, $d(\lambda, x) = 2\sqrt{1 + \lambda^2 - \lambda(x+x^{-1})}$, and $w = (\lambda_i + \lambda_f)/(2\pi\lambda_f\sqrt{\lambda_i\Lambda_i})$. Accordingly, the susceptibility is $\chi_i(\lambda_i, \lambda_f, 0) = \int_0^{\Lambda_i} dx [\pi x d(\lambda_i, x)]^{-1} s(\lambda_i, x) u + C(\lambda_i, \lambda_f)$, where $u = \text{sgn}(\Lambda_f - x)c(\lambda_f, x)[s(\lambda_f, x)c(\lambda_i, x) - c(\lambda_f, x)s(\lambda_i, x)]$ and $C(\lambda_i, \lambda_f)$ is convergent at $\lambda_i=1$. In the case of $\Lambda_i \approx 1$, i.e., $\lambda_i \sim 1$, the above integral can be reduced as $\int_0^{\Lambda_i} (\dots) dx \approx \int_{\Lambda_i - \epsilon}^{\Lambda_i} (\dots) dx$ with $\epsilon \ll 1$. Via straightforward calculations, an asymptotic behavior of $\chi_i(\lambda_i, \lambda_f, 0)$ around $\lambda_i=1$ is obtained as

$$\chi_i(\lambda_i, \lambda_f, 0) \approx -\frac{1}{\pi} \ln|\lambda_i - 1| + K_1(\lambda_f), \quad (7)$$

where $K_1(\lambda_f)$ is a λ_f -dependent constant. In Figs. 1(a) and 1(b), $\chi_i(\lambda_i, \lambda_f, 0)$ are plotted as functions of λ_i and $\ln|\lambda_i - 1|$ with $\lambda_f=0, 0.6, 1.5$, and ∞ .

According to the finite-size scaling ansatz,¹⁷ the above critical behavior can be extracted from finite samples, which

is important for quantum simulations. Numerical simulations for finite systems show that the susceptibility reaches the maximum $\chi_i(\lambda_m, \lambda_f, 0)$ at the pseudocritical-point λ_m . In Fig. 1(c), we plot $\chi_i(\lambda_m, \lambda_f, 0)$ as a function of $\ln N$ with $\lambda_f=0, 0.6, 1.5$, and ∞ . As expected, $\chi_i(\lambda_m, \lambda_f, 0)$ diverges logarithmically as

$$\chi_i(\lambda_m, \lambda_f, 0) \approx \frac{1}{\pi} \ln N + K_2(\lambda_f), \quad (8)$$

where $K_2(\lambda_f)$ is another λ_f -dependent constant. As the lattice size approaches infinite, the pseudocritical-point λ_m tends to the QCP as $|\lambda_m - 1| \propto N^{-1.9}$, which is shown in Fig. 1(d). According to the scaling ansatz in the case of logarithmic divergence,¹⁷ the ratio between the two prefactors of the logarithm in Eqs. (7) and (8) is the exponent ν that governs the divergence of the correlation length. As expressed in Eqs. (7) and (8), numerical calculations give $\nu=1$, which accords with the result obtained from the exact solution of the TFIM.¹⁷

When the initial system is at finite temperature, the correlation length of $\rho(\lambda_i, T, 0)$ is always convergent and all the scaling behaviors will vanish. We do not discuss this case in detail.

IV. DISCONTINUITY OF THE QUENCH SUSCEPTIBILITY

Now we focus on $m_q(\lambda_i, \lambda_f, T)$ and $\chi_f(\lambda_i, \lambda_f, T)$ at $\lambda_f \sim 1$. Mathematically, under the condition $\Lambda_f > \Lambda_i$, Eq. (6) becomes

$$m_q(\lambda_i, \lambda_f, T) = \frac{|1 - \lambda_f^2|}{4\lambda_f^2} \tanh \Theta [c(\lambda_i, \Lambda_f) - s(\lambda_i, \Lambda_f)] + A(\lambda_i, \lambda_f, T), \quad (9)$$

where $\Theta = d(\lambda_i, \Lambda_f)/2T$ and $A(\lambda_i, \lambda_f, T)$ is an analytical function when λ_i is away from 1.¹⁶ Obviously, the factor $|1 - \lambda_f^2|$ in Eq. (9) causes a sudden change of $m_q(\lambda_i, \lambda_f, T)$ at $\lambda_f = 1$. For two extreme cases, $\lambda_i = 0$ and ∞ , Eq. (6) is completely integrable and could be written as $m_q(0, \lambda_f, T) = \tanh(1/T)(4\lambda_f)^{-1}[(1 + \lambda_f^2) + \text{sgn}(\lambda_f - 1)(1 - \lambda_f^2)]$ and $m_q(\infty, \lambda_f, T) = [(3 - \lambda_f^2) + \text{sgn}(\lambda_f - 1)(1 - \lambda_f^2)]/4$.

Accordingly, such a sudden change leads to a discontinuity of $\chi_f(\lambda_i, \lambda_f, T)$ at the QCP. The expression is

$$\chi_f(\lambda_i, \lambda_f, T) = \text{sgn}(\lambda_f - 1) \frac{\tanh \Theta}{2\lambda_f^3} [c(\lambda_i, \Lambda_f) - s(\lambda_i, \Lambda_f)] + B(\lambda_i, \lambda_f, T), \quad (10)$$

where $B(\lambda_i, \lambda_f, T)$ is continuous at $\lambda_f \sim 1$. The sgn function

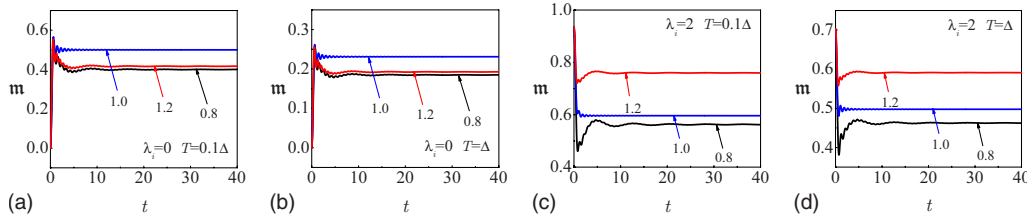


FIG. 3. (Color online) The time evolution of $m(\lambda_i, \lambda_f, T, t)$ with $\lambda_i=0, 2$; $T/\Delta_i=0.1, 1$, and $\lambda_f=0.8, 1, 1.2$. $m(\lambda_i, \lambda_f, T, t)$ oscillates initially and becomes steady at $t=15$, approximately.

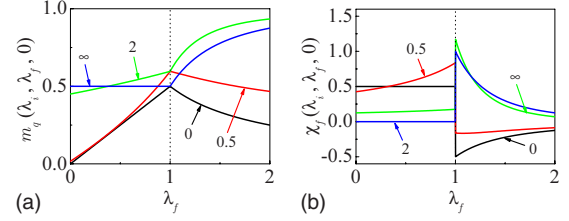


FIG. 2. (Color online) Plots of $m_q(\lambda_i, \lambda_f, 0)$ and $\chi_f(\lambda_i, \lambda_f, 0)$ with $\lambda_i=0, 0.5, 2$, and ∞ . $m_q(\lambda_i, \lambda_f, 0)$ has a sudden change at $\lambda_f = 1$, while $\chi_f(\lambda_i, \lambda_f, 0)$ has a discontinuity at $\lambda_f=1$. When λ_f passes through 1, $\chi_f(\lambda_i, \lambda_f, 0)$ drops down for $\lambda_i < 1$ and jumps up for $\lambda_i > 1$.

leads to a jump of $\chi_f(\lambda_i, \lambda_f, T)$ with magnitude $\delta\chi_f = \tanh(\Delta_i/2T)$, where $\Delta_i = 2|\lambda_i - 1|$ is the energy gap above the ground state for the initial Hamiltonian $H(\lambda_i)$. Remarkably, the jump $\delta\chi_f$ always occurs at the QCP even for a finite-temperature initial state. When $T \leq \Delta_i$, $\delta\chi_f$ decays slowly as $1 - 2 \exp(-\Delta_i/T)$, and is almost a constant within the range $T < 0.1\Delta_i$. In this sense, the lower-temperature samples share the same feature as that of the zero-temperature sample, which is crucial for the experimental detection of critical behaviors of QPT systems, since cooling matter to zero temperature is impossible in any experiment.

To exhibit the above critical behaviors, in Fig. 2, $m_q(\lambda_i, \lambda_f, 0)$ and $\chi_f(\lambda_i, \lambda_f, 0)$ with $\lambda_i=0, 0.5, 2$, and ∞ are plotted. It is shown that $m_q(\lambda_i, \lambda_f, 0)$ has a sudden change for systems with a wide range of λ_i .

It is well-known that the ground state of the system $H(\lambda_f)$ experiences a sudden change when λ_f passes through the QCP. A natural question is whether the sudden change of the ground state causes the critical behaviors of quench quantities directly. Consider the simplest case with zero-temperature initial state, where the jump $\delta\chi_f$ gets the maximum 1. The contribution of the final ground state $|\theta(\lambda_f)\rangle$ to the quench quantity is proportional to the fidelity of two ground states^{18,19}

$$|\langle \theta(\lambda_i) | \theta(\lambda_f) \rangle|^2 = \prod_k \cos^2 \frac{1}{2} [\theta_k(\lambda_f) - \theta_k(\lambda_i)]. \quad (11)$$

Straightforward calculations show that $|\langle \theta(\lambda_i) | \theta(\lambda_f) \rangle|$ is always vanishing for finite $|\lambda_f - \lambda_i|$ in the thermodynamic limit. Thus, the critical behaviors of quench quantities are not direct consequences of the sudden change of the ground state, which indicates that the excited states also experience drastic changes at the QCP.

V. DISCUSSION AND CONCLUSION

In above analyses, we neglect the interaction between the sample of TFIM and its environment. Generally, such an interaction will induce decoherence of the system. Therefore, our results are valid only when the quench relaxation-time τ_Q is short enough compared to the decoherence time. In order to estimate the order of τ_Q , numerical simulations for $m(\lambda_i, \lambda_f, T; t)$ are performed. In Fig. 3, we plot the results with $\lambda_i=0, 2$, $T/\Delta_i=0.1, 1$, and $\lambda_f=0.8, 1, 1.2$ as examples. Remarkably, τ_Q is not sensitive to T , λ_i , and λ_f , and τ_Q is smaller than $15J^{-1}$, where J is the Ising coupling strength and in this paper, $J=1$. In a spin network for quantum information processing (QIP), the shortest period of time for an operation between two neighbor qubits via a natural dynamics is $(2J)^{-1}$. Therefore the realization of a complete quench process is not a difficult task for a spin network which is utilized for QIP in practice.

In summary, we found that the quench magnetic susceptibility as a function of the initial field strength exhibits

strongly similar scaling behaviors to those of the adiabatic process, and the quench magnetic susceptibility as a function of the final field strength shows a discontinuity at the QCP, which remains robust even when the initial system is at finite temperature. This observation is useful for understanding QPTs and studying the properties of the QPT systems. Moreover, it gives us an approach to observe the QCP and critical behaviors using low-temperature samples experimentally, avoiding the rigorous restriction of zero temperature. Because of the universality principle, the critical behaviors only depend on the dimension and the breaking symmetry, so the obtained results are heuristic and may be extended to other QPT models.

ACKNOWLEDGMENT

This work was supported by the CNSF with Grants No. 10874091 and No. 2006CB921205.

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¹⁶The integral in Eq. (6) can be written as a circular integral in a complex plane via the transformation $z=e^{ik}$. For $\Lambda_f > \Lambda_i$, there is a branch cut along the real axis from 0 to Λ_i . For $\Lambda_f < \Lambda_i$, besides the branch cut, there is an isolated singularity at $z=\Lambda_f$.

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